Return Models - Part II
The Stochastic, Mean-Reverting Short Rate

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Economic rates such as the rate of interest, the rate of revenue growth, and the rate of return on investment exhibit mean reversion, which is the tendency for a stochastic process to return over time to a long-term mean. We will define the variable r_t to be the stochastic, annualized short rate at time t. We will model the short rate as an Ornstein-Uhlenbeck process, which is a mean-reverting process where the short-term rate (unsustainable) is allowed to incorporate random shocks but is pulled back to its long-term mean (sustainable) whenever it moves away from it. The stochastic differential equation that describes the evolution of the short rate is... [1]

$$\delta r_t = \lambda \Bigg(r_\infty - r_t \Bigg) \delta t + \sigma \, \delta W_t \;\; \text{...where...} \;\; 0 < \lambda < 1 \tag{1}$$

In the equation above, when the short rate (r_t) moves below the long-term mean (r_∞) drift becomes positive and the short rate is pulled upward. When the short rate moves above the long-term mean drift becomes negative and the short rate is pulled downward. The speed at which the drift is pulled upward of downward is given by the positive valued parameter λ, which measures the speed of mean reversion. The greater the speed the faster the process reverts towards the long-term mean. Random shocks are introduced via the variable σ, which is annualized short rate volatility, and the variable δW_t, which is the change in the driving Brownian motion over the infinitesimally small time interval $[t, t + \delta t]$.

The short rate at time t is equal to the short rate at time zero plus the cumulative changes to the short rate that occur over the time interval $[0, t]$. Using Equation (1) above the equation for the short rate at time t is...

$$r_t = r_0 + \int_0^t \delta r_u \tag{2}$$

In this white paper we will develop the mathematics to model the stochastic, mean-reverting short rate. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

ABC Company has a revenue growth rate that is currently well above the growth rate of nominal GDP. We know that the revenue growth rate will decrease from the short-term unsustainable rate to the long-term sustainable rate over time. We are tasked with modeling ABC Company revenue over time. We are given the following model parameters...

Table 1: Go-Forward Assumptions

Description	Balance
Annualized revenue at time zero ($)	800,000
Revenue growth rate at time zero (%)	20.00
Revenue growth long-term sustainable rate (%)	5.00
Revenue growth rate standard deviation (%)	9.50
Transition half-life in years (#)	7.00

We are tasked with answering the following questions...

Question 1: What is the spot revenue growth rate at the end of year 5?

Question 2: What is the cumulative revenue growth rate at the end of year 5?

Question 3: What is expected annualized revenue at the end of year 5?

Question 4: What is probability that actual annualized revenue at the end of year 5 will be 1.5 x the expectation?

Short Rate Distribution Mean And Variance

The equation for the first moment of the distribution of the short rate at time t is... [1]

$$\mathbb{E}\left[r_t\right] = r_0 \operatorname{Exp}\left\{-\lambda\, t\right\} + r_\infty\left(1 - \operatorname{Exp}\left\{-\lambda\, t\right\}\right) \tag{3}$$

The equation for the second moment of the distribution of the short rate at time t is... [1]

$$\mathbb{E}\left[r_t^2\right] = \mathbb{E}\left[r_t\right]^2 + \frac{\sigma^2}{2\,\lambda}\left(1 - \operatorname{Exp}\left\{-2\,\lambda\, t\right\}\right) \tag{4}$$

Using Equation (3) above the mean of the short rate at time t is...

$$\begin{aligned} \text{mean} &= \mathbb{E}\left[r_t\right] \\ &= r_0 \operatorname{Exp}\left\{-\lambda\, t\right\} + r_\infty\left(1 - \operatorname{Exp}\left\{-\lambda\, t\right\}\right) \\ &= r_\infty + \left(r_0 - r_\infty\right)\operatorname{Exp}\left\{-\lambda\, t\right\} \end{aligned} \tag{5}$$

Using Equation (5) above the equation for the expected short rate at time zero is...

$$\mathbb{E}\left[r_0\right] = r_\infty + \left(r_0 - r_\infty\right)\operatorname{Exp}\left\{-\lambda \times 0\right\} = r_0 \text{ ...because... } \operatorname{Exp}\left\{-\lambda \times 0\right\} = 1 \tag{6}$$

Using Equation (5) above the equation for the expected short rate at time infinity is...

$$\mathbb{E}\left[r_\infty\right] = r_\infty + \left(r_0 - r_\infty\right)\operatorname{Exp}\left\{-\lambda \times \infty\right\} = r_\infty \text{ ...because... } \operatorname{Exp}\left\{-\lambda \times \infty\right\} = 0 \tag{7}$$

Using Equations (3) and (4) above the variance of the short rate at time t is...

$$\text{variance} = \mathbb{E}\left[r_t^2\right] - \mathbb{E}\left[r_t\right]^2 = \frac{1}{2}\,\sigma^2\left(1 - \operatorname{Exp}\left\{-2\,\lambda\, t\right\}\right)\lambda^{-1} \tag{8}$$

Simulating The Short Rate

Given that r_t is the stochastic short rate at time t (unknown) given the short rate at time zero (known), m is the short rate **mean** per Equation (5) above and v is the short rate **variance** per Equation (8) above, we can define the normalized random variable Z via the following equation...

$$\frac{r_t - m}{\sqrt{v}} = Z \text{ ...where... } Z \sim N\left[0, 1\right] \tag{9}$$

By rearranging Equation (9) above the simulated value of the short rate at some future time t given the short rate at time zero is...

$$r_t = m + \sqrt{v}\, Z \text{ ...where... } Z \sim N\left[0, 1\right] \tag{10}$$

Cumulative Short Rate Distribution Mean And Variance

We will define the variable Γ_t to be the cumulative short rate over the time interval $[0, t]$. Using Equations (1) and (2) above the equation for the cumulative rate is...

$$\Gamma_t = \int\limits_0^t r_u \, \delta u \tag{11}$$

The equation for the first moment of the distribution of the cumulative rate at time t is... [1]

$$\mathbb{E}\bigg[\Gamma_t\bigg] = r_\infty \, t + \bigg(r_\infty - r_0\bigg)\bigg(\mathrm{Exp}\bigg\{-\lambda \, t\bigg\} - 1\bigg)\lambda^{-1} \tag{12}$$

The equation for the second moment of the distribution of the cumulative rate at time t is... [1]

$$\mathbb{E}\bigg[\Gamma_t^2\bigg] = \mathbb{E}\bigg[\Gamma_t\bigg]^2 + \frac{1}{2}\sigma^2\bigg(2\,\lambda\,t - 3 + 4\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} - \mathrm{Exp}\bigg\{-2\,\lambda\,t\bigg\}\bigg)\lambda^{-3} \tag{13}$$

Using Equation (12) above the mean of the distribution of the cumulative rate at time t is...

$$\mathrm{mean} = \mathbb{E}\bigg[\Gamma_t\bigg] = r_\infty \, t + \bigg(r_\infty - r_0\bigg)\bigg(\mathrm{Exp}\bigg\{-\lambda \, t\bigg\} - 1\bigg)\lambda^{-1} \tag{14}$$

Using Equations (12) and (13) above the variance of the distribution of the cumuative rate at time t is...

$$\mathrm{variance} = \mathbb{E}\bigg[\Gamma_t^2\bigg] - \mathbb{E}\bigg[\Gamma_t\bigg]^2 = \frac{1}{2}\sigma^2\bigg(2\,\lambda\,t - 3 + 4\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} - \mathrm{Exp}\bigg\{-2\,\lambda\,t\bigg\}\bigg)\lambda^{-3} \tag{15}$$

We will need the derivative of the cumulative rate distribution mean with the respect to time. Using Equation (14) above the equation for the derivative of the cumulative rate mean is...

$$\frac{\delta}{\delta t}\,\mathbb{E}\bigg[\Gamma_t\bigg] = r_\infty + \bigg(r_0 - r_\infty\bigg)\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} \tag{16}$$

Simulating The Cumulative Short Rate

Given that Γ_t is the stochastic cumulative short rate at time t (unknown) given the short rate at time zero (known), m is the cumulative short rate **mean** per Equation (14) above and v is the cumulative short rate **variance** per Equation (15) above, we can define the normalized random variable Z via the following equation...

$$\frac{\Gamma_t - m}{\sqrt{v}} = Z \;\; \text{...where...} \;\; Z \sim N\bigg[0, 1\bigg] \tag{17}$$

By rearranging Equation (17) above the simulated value of the cumulative short rate at some future time t given the short rate at time zero is...

$$\Gamma_t = m + \sqrt{v}\,Z \;\; \text{...where...} \;\; Z \sim N\bigg[0, 1\bigg] \tag{18}$$

Annualized Revenue

We will define the variable R_t to be annualized revenue at time t. Using Equation (11) above the equation for expected annualized revenue (i.e the mean) at time t is...

$$R_t = R_0\,\mathrm{Exp}\bigg\{\Gamma_t\bigg\} \tag{19}$$

Using Equation (14) above we can rewrite Equation (19) above as...

$$R_t = R_0\,\mathrm{Exp}\bigg\{r_\infty \, t + \bigg(r_\infty - r_0\bigg)\bigg(\mathrm{Exp}\bigg\{-\lambda \, t\bigg\} - 1\bigg)\lambda^{-1}\bigg\} \tag{20}$$

We will define the variable $R_{0,t}$ to be cumulative revenue realized over the time interval $[0,t]$. Using Equation (20) above

$$R_{0,t}=\int\limits_0^t R_u\,\delta u=R_0\int\limits_0^t \mathrm{Exp}\left\{r_\infty\,u+\left(r_\infty-r_0\right)\left(\mathrm{Exp}\left\{-\lambda\,u\right\}-1\right)\lambda^{-1}\right\}\delta u \tag{21}$$

Note that in the equation above we have an exponential within an exponential such that the antiderivative of the integrand cannot be calculated using standard methods. To solve that integral we must use either the exponential integral or the incomplete gamma function.

Simulating Annualized Revenue

Note that per Equation (18) above the random cumulative short rate Γ_t is normally-distributed with mean m and variance v. Note also that per Equation (19) above random annualized revenue, with is the exponential of the normally-distributed short rate, is lognormally-distributed. We can make the following statements...

$$\text{If... }\Gamma_t\sim N\left[m,v\right]\text{ ...then... }\mathbb{E}\left[R_t\right]=R_0\,\mathrm{Exp}\left\{m+\frac{1}{2}\,v\right\} \tag{22}$$

We will define the variable $\bar{R}_t$ to be the random (i.e. simulated) value of annualized revenue at time t. Using Equation (22) above we can simulate random annualized revenue via the following equation...

$$\text{If... }\bar{R}_t=R_0\,\mathrm{Exp}\left\{m-\frac{1}{2}\,v+\sqrt{v}\,Z\right\}\text{ ...then... }\mathbb{E}\left[\bar{R}_t\right]=R_0\,\mathrm{Exp}\left\{m\right\} \tag{23}$$

Note that expected annualized revenue per Equation (23) above is now consistent with expected annualized revenue per Equation (19) above.

Using Equation (23) above the value of the normally-distributed random variable Z that gives us random annualized revenue at time t of $\bar{R}_t$ is...

$$\begin{aligned}\bar{R}_t&=R_0\,\mathrm{Exp}\left\{m-\frac{1}{2}\,v+\sqrt{v}\,Z\right\}\\ \ln(\bar{R}_t)-\ln(R_0)&=m-\frac{1}{2}\,v+\sqrt{v}\,Z\\ Z&=\left(\ln(\bar{R}_t)-\ln(R_0)-m+\frac{1}{2}\,v\right)\Big/\sqrt{v}\end{aligned} \tag{24}$$

Using Equation (24) above the the probability that actual annualized revenue at time t exceeds a given value of R_t is...

$$\mathrm{Prob}\left[\text{Actual annualized revenue at time } t>\bar{R}_t\right]=1-\mathrm{CNDF}(Z)\text{ ...where... }Z\sim N\left[0,1\right] \tag{25}$$

Note that function CNDF(Z) in Equation (24) above is the cumulative normal distribution function with mean zero and variance one.

Calibrating The Rate Of Mean Reversion

Using Equations (6) and (7) above the short rate transitions from r_0 to r_∞ over the time interval $[0,\infty]$. The rate of mean reversion (λ) has a value between zero and one such that the higher the value the faster the short-term unsustainable rate transitions to the long-term sustainable rate. To calibrate the rate of mean reversion we will choose some future point in time ($t=T$) where the expected short rate at time T is halfway between the short-term rate and the long-term rate (i.e. the half life). Using Equation (5) above the equation to calibrate the rate of mean reversion is...

$$\mathrm{Exp}\left\{-\lambda\times T\right\}=0.50\text{ ...such that... }\lambda=-\frac{\ln(0.50)}{T} \tag{26}$$

The Answers To Our Hypothetical Problem

Using Equation (6) above and the model assumptions in Table 1 above the continuous-time model parameters are...

Table 2: Model Parameters

Description	Symbol	Value	Calculation
Annualized revenue at time zero	R_0	800,000	
Short-term revenue growth rate	r_0	0.1823	$\ln(1+0.20)$
Long-term revenue growth rate	r_∞	0.0488	$\ln(1+0.05)$
Revenue growth rate standard deviation	σ	0.0908	$\ln(1+0.095)$
Transition rate	λ	0.0990	$-\ln(0.5) \div 7$

Question 1: What is the spot revenue growth rate at the end of year 5?

Using Equation (3) the expected spot revenue growth rate at the end of year 5 is...

$$\mathbb{E}\left[r_5\right] = 0.1823 \times \text{Exp}\left\{-0.0990 \times 5\right\} + 0.0488\left(1 - \text{Exp}\left\{-0.0990 \times 5\right\}\right) = 0.1302 \tag{27}$$

Question 2: What is the cumulative revenue growth rate at the end of year 5?

Using Equation (14) the expected cumulative revenue growth rate at the end of year 5 is...

$$\mathbb{E}\left[\Gamma_5\right] = 0.0488 \times 5 + \left(0.0488 - 0.1823\right) \times \left(\text{Exp}\left\{-0.0990 \times 5\right\} - 1\right) \times 0.0990^{-1} = 0.7705 \tag{28}$$

Question 3: What is expected annualized revenue at the end of year 5?

Using Equations (19) and (28) above expected annualized revenue at the end of year 5 is...

$$R_t = 800,000 \times \text{Exp}\left\{0.7705\right\} = 1,728,677 \tag{29}$$

Question 4: What is probability that actual annualized revenue at the end of year 5 will be 1.5 x the expectation?

Using Equation (28) above the cumulative revenue growth rate mean at the end of year 5 is...

$$m = 0.7705 \tag{30}$$

Using Equation (15) above the cumulative revenue growth rate variance at the end of year 5 is...

$$v = \frac{1}{2} \times 0.0908^2 \times \left(2 \times 0.0990 \times 5 - 3 + 4\,\text{Exp}\left\{-0.0990 \times 5\right\} - \text{Exp}\left\{-2 \times 0.0990 \times 5\right\}\right) \times 0.0990^{-3} = 0.2407 \tag{31}$$

Using Equations (25), (29), (30) and (31) above the value of the random variable Z that gives us random annualizsed revenue that is equal to 1.5 times expected annualized revenue is...

$$Z = \left(\ln(1,728,677 \times 1.5) - \ln(800,000) - 0.7705 + \frac{1}{2} \times 0.2407\right) \Big/ \sqrt{0.2407} = 1.0718 \tag{32}$$

Using Equations (25) and (32) above the probability that actual annualized revenue will be greater than 1.5 times expected annualized revenue is...

$$\text{Prob}\left[\text{Actual annualized revenue at end of year 5} > 1,728,677 \times 1.5\right] = 1 - \text{CNDF}(1.0718) = 0.1419 \tag{33}$$

References

[1] Gary Schurman, *The Vasicek Interest Rate Process - Part I,* February, 2013.